MOLDOVA STATE UNIVERSITY

DOCTORAL SCHOOL OF NATURAL SCIENCES

With manuscript title
C.Z.U.: 512.548 (043.2)

CUZNEȚOV ELENA

EXTENSIBILITY AND RECURSIVE DIFFERENTIABILITY OF QUASIGROUPS

111.03. Mathematical logic, algebra and number theory

Abstract of the doctoral thesis in mathematical sciences

Chișinău, 2026

The thesis was elaborated within the Moldova State University, Doctoral School of Natural Sciences, Department of Mathematics.

Scientific supervisor:

SÎRBU Parascovia	PhD in Physical and Mathematical Sciences, Associate Professor, Moldova State University

Public Defense Committee for the doctoral thesis:

RUSU Andrei	PhD in Physical and Mathematical Sciences, Associate Professor, "V. Andrunachievici" Institute of Mathematics and Computer Science, Moldova State University – ***Chair;***
SÎRBU Parascovia	PhD in Physical and Mathematical Sciences, Associate Professor, Moldova State University **- *scientific supervisor;***
IZBAȘ Vladimir	PhD in Physical and Mathematical Sciences, Associate Professor, "V.Andrunachievici" Institute of Mathematics and Computer Science, Moldova State University - ***official reviewer;***
SOKHATSKY Fedir	PhD habilitate in Physical and Mathematical Sciences, Professor, Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NASU, Ukraine – ***official reviewer;***
CHIRIAC Liubomir	PhD habilitate in Physical and Mathematical Sciences, Professor, Ion Creangă State Pedagogical University of Chișinău - ***official reviewer.***

The defense will take place on 23.06.2026, at 10^{00}, at the meeting of the Public Defense Committee for the doctoral theses within the Doctoral School of Natural Sciences, at the Moldova State University, at the following address: Republic of Moldova, Chișinău, MD-2009, 60 A. Mateevici Street, building 4, room 222.

The Doctoral Thesis and the Abstract can be consulted at the National Library of the Republic of Moldova, the Central Library of the Moldova State University (MD-2009, Chișinău, A. Mateevici str., 60) and on the National Agency for Quality Assurance in Education and Research (ANACEC) website (http://www.anacec.md)

The abstract was sent on 08.05. 2026

Chair of the Public Defense Committee for Doctoral Thesis: PhD in physical and mathematical sciences, associate professor	**RUSU Andrei**
Scientific supervisor: PhD in physical and mathematical sciences, associate professor	**SÎRBU Parascovia**
Author:	**CUZNEȚOV Elena**

CONTENTS

1. CONCEPTUAL GUIDELINES OF THE RESEARCH

The actuality and importance of the research topic. The development of quasigroup theory began in the 1930s. The term *quasigroup* was introduced in a paper published by R. Moufang in 1935, which referred to the coordination of projective planes. However, the concept of quasigroups had actually been considered much earlier by Schroeder, who, between 1873 and 1890, wrote a series of papers on "formal arithmetic": i.e., algebraic structures with a binary operation such that both the left inverse and the right inverse could be uniquely defined. Such a structure is a quasigroup [3, 9, 23].

A binary groupoid (Q, A) is called a quasigroup if, for $\forall a, b \in Q$, each of the equations $A(a, x) = b$ and $A(y, a) = b$ have a unique solution in Q .

Latin squares are the combinatorial analogue of finite binary quasigroups. We call a latin square of order q, defined on a set Q of q elements, any table with q rows and q columns, at the intersection of which are the elements of the set Q, such that the elements do not repeat either in rows or in columns. Quasigroups (latin squares) have numerous practical applications. In particular, due to their combinatorial properties, they are used in coding theory and cryptography, in the theory of experimental design, in automata theory, etc.

Let Q be a finite set of q elements. Any non-empty subset $C \subseteq Q^n$, where $n \geq 1$, is called a code of length n, or a n -code, over the alphabet Q. An n-code $C \subseteq Q^n$ is an $[n, k]_q$-code if $|C| = q^k$, where $|Q| = q$ and $k \geq 1$, and an $[n, k]_q$-code C is an $[n, k, d]_q$-code if its minimum Hamming distance of C is d. It is known that the parameters of an $[n, k, d]_q$-code satisfy the inequality $d \leq n - k + 1$ [17, 18, 27]. If an $[n, k, d]_q$-code has the minimum Hamming distance equal to $n - k + 1$, then we say that this code reaches the upper Singleton bound, i.e. is an MDS-code. It currently remains an open problem to determine all sets (n, k, d, q) of natural numbers such that there exist MDS codes C of length n, over an alphabet of q elements, with $|C| = q^k$ and with the minimum Hamming distance d.

A code $C \subseteq Q^n$ is called a complete s-recursive code, where $1 \leq s \leq n - 1$, if there exists a mapping $f: Q^s \to Q$, such that any code word $u = (u_0, u_1, \dots, u_{n-1}) \in C$ satisfies the condition $u_{i+s} = f(u_i, u_{i+1}, \dots, u_{i+s-1})$, for any $i = 0,1, \dots, n - s - 1$, denoting in this case $C = C(n, f)$.

The notions of "recursive derivative" and "recursively differentiable quasigroup" were introduced in [10], where the authors study complete recursive MDS codes. The recursive derivative of order $t \geq 0$ of an k-ary groupoid (Q, A) is denoted by $A^{(t)}$ and is defined as follows:

$$A^{(0)} = A,$$

$$A^{(t)}(x_1^k) = A(x_{t+1}, \dots, x_k, A^{(0)}(x_1^k), \dots, A^{(t-1)}(x_1^k)) \text{ if } 1 \leq t < k;$$

$$A^{(t)}(x_1^k) = A(A^{(t-k)}(x_1^k), \ldots, A^{(t-1)}(x_1^k)) \text{ if } t \geq k,$$

$\forall x_1, \ldots, x_k \in Q$, where we denote the sequence $x_1, x_2, \ldots, x_k$ by x_1^k.

An n-ary grupoid (Q, A) is called an n -ary quasigroup, or n-quasigroup, if in the equality $A(x_1, \ldots, x_n) = x_{n+1}$ any n of the elements $x_1, \ldots, x_n, x_{n+1}$ uniquely determine the remaining one [4]. A k-ary quasigrup (Q, A) is called *recursively r-differentiable* if its recursive derivatives $A^{(0)}, A^{(1)}, \ldots, A^{(r)}$ are quasigroup operations ($r \geq 0$).

In the case of a binary quasigroup $(Q,*)$, denoting by $\overset{t}{*}$ the recursive derivative of order t of the operation $*$, we have:

$$x \overset{0}{*} y = x * y,$$

$$x \overset{1}{*} y = y * (x * y),$$

$$x \overset{t}{*} y = (x \overset{t-2}{*} y) * (x \overset{t-1}{*} y), \forall t \geq 2 \text{ and } \forall x, y \in Q.$$

It is known that there exist recursively 1-differentiable finite binary quasigroups of any order q, exept for $q = 1, 2, 6$ and possibly $14, 18, 26$ [10, 21]. Some estimations of the maximum order of recursive differentiability of finite n quasigroups ($n \geq 2$) are given in [1, 10]. General properties of recursively differentiable binary quasigroups are studied in [8, 10, 15, 16, 19].

The length n of code words in a k-recursive code

$$C(n, A) = \{(x_1, \ldots, x_k, A^{(0)}(x_1^k), \ldots, A^{(n-k-1)}(x_1^k)) | x_1, \ldots, x_k \in Q\},$$

defined over an alphabet Q of q elements, where $A: Q^k \to Q$ is a k-ary operation of quasigroup, satisfies the inequality $n \leq r + k - 1$, where r is the maximum order of recursive differentiability of the quasigroup (Q, A). The study of the parameters for which $C(n, A)$ is an MDS-code implies, in particular, the problem of determining the maximum order of recursive differentiability of k-ary finite quasigroups ($k \geq 2$).

The recursive differentiability of n-ary quasigroups is closely related to the orthogonality of recursive derivatives [10]. n-Ary groupoids $(Q, A_1), \ldots, (Q, A_{\mathrm{n}})$ are called orthogonal if, for any $a_1, \ldots, a_n \in Q$, the system of equations

$$\begin{cases} A_1(x_1, \ldots, x_n) = a_1 \\ \quad \ldots \ldots \ldots \ldots \\ A_n(x_1, \ldots, x_n) = a_n \end{cases}$$

has a unique solution in Q. A set of t n-ary operations defined on a set Q, where $t \geq n$, is called an orthogonal system if any n operations of this set are orthogonal [17, 18]. A system of n-ary operations $A_1, \ldots, A_t$, where $t \geq 1$, defined on a set Q, is called a strongly orthogonal system if the system $\{A_1, \ldots, A_t, E_1, \ldots, E_n\}$ is orthogonal, where $E_i(x_1, \ldots, x_n) = x_i$, for any $x_1, \ldots, x_n \in Q$, is the n-ary i - selector, $i \in \{1,2, \ldots, n\}$.

In [10] it is shown that a k-quasigroup defines an MDS code of length n if and only if its first $n-k-1$ recursive derivatives form a strongly orthogonal system. Therefore, the operation of the k-quasigroup that defines a recursive MDS code of length n is recursively $(n-k-1)$ – differentiable. Also, it is known that a system of binary quasigroups is strongly orthogonal if and only if it is (simply) orthogonal [17, 18].

Another "special property" of binary quasigroups is given in [10]: recursive derivatives of order up to r of a finite binary quasigroup $(Q,*)$ are quasigroup operations if and only if $(Q,*)$ defines a recursive MDS code of length $r+3$. Thus, a finite binary quasigroup $(Q,*)$ is recursively r-differentiable if and only if the set of its recursive derivatives of order s, where s$\leq r$, form an orthogonal system. The last statement implies that there does not exist recursively 1-differentiable quasigroups of order 2 or 6, and that r can not exeed $q-2$, where $q=|Q|$ and r is the order of recursive differentiability of the quasigroup Q. An algebraic proof of the inequality $r \leq q-2$ is given in [24].

In [10] it is shown that there exist finite binary recursively $(q-2)$-differentiable quasigroups of any primary order $q \geq 3$. However, it remains an open problem to determine the maximum order r of recursive differentianility of finite k-ary quasigroups of order q, for $k \geq 2$ and q non-primary.

In the theory of orthogonal operations (orthogonal latin squares), the notion of transversal plays an important role [3, 17, 18]. We call a transversal of a latin square of order n a set of n cells taken by one from each row and each column of the latin square, such that the elements in these cells are pairwise distinct. It is known that for a latin square L of order n, there exists an orthogonal latin square if and only if L decomposes into n pairwise disjoint transversals.

The notion of transversal is closely related to that of complete mapping (or complete substitution) [3, 13, 14, 18, 22, 25, 26]. A bijective fuction $\sigma \in S_Q$ is called a complete mapping of a quasigroup $(Q,\cdot)$ if the function $\sigma': Q \rightarrow Q,\ \sigma'(x) = x \cdot \sigma(x)$ is bijective. The concept of complete mapping was introduced by Mann [20]. If Q is a quasigroup that has a complete mapping, then its multiplication table is a latin square with a transversal. Conversely, if L is a latin square that has a transversal, then the quasigroup, that has L as its multiplication table, has a complete mapping. Any finite group of odd order possesses a complete mapping: in these groups, the function $x \rightarrow xx$ is bijective [18].

In 1967 Ryser raised the question of whether there are quasigroups of odd order that do not have a complete mapping. It is known that there are quasigroups of even order that do not have complete mappings. In particular, Mann proved that, if the quasigroup Q of order $4k+2$

possesses a subquasigroup of order $2k + 1$, then its multiplicative table has no transversals. It is relevant to note that if Q is a quasigroup that has a complete mapping, then any isotope of Q has a complete mapping as well [17, 18].

A prolongation of a finite quasigroup Q is a higher-order quasigroup obtained from Q by adjoining a finite number of new elements and redefining the operation. The notion of prolongation was introduced by V. Belousov in 1967, who, using a finite quasigroup of order n with a complete mapping, presented a method for constructing a quasigroup of order $n + 1$, by adjoining a new element [2, 3]. This method generalizes a method given by Bruck (1944) for extending a finite idempotent quasigroup by adjoining a new element. Methods of prolongation of quasigroups were also given by Osborn (1961), Denes and Pasztor (1963), Belyavskaya (1969), Elspas, Minnick, and Short (1963), Deriyenko and Dudek (2008, 2013), and Derienko (2009) [7, 12]. Remark that Yamamoto (1961) used the same concept, calling it 1-extension, in connection with the construction of pairs of mutually orthogonal latin squares. He also defined an inverse construction, which he called 1-contraction, subsequently studied in a series of papers [5, 6, 11, 28].

The purpose and objectives of the work. The purpose of the thesis is to characterize the recursive differentiability of binary and n-ary quasigroups, including prolongations of quasigroups, to estimate the maximum order of recursive differentiability and the spectrum of finite recursively differentiable quasigroups. To achieve the intended goal, the following objectives are set: characterizing the recursive differentiability of binary quasigroups and n-ary groups, to determine new methods of prolongation of finite binary quasigroups, and to study the recursive differentiability of quasigroup prolongations.

Novelty and scientific originality In the present work, criteria for recursive differentiability of order $n \geq 1$ of binary and n-ary quasigroups are determined, a new method of prolongation of finite binary quasigroups is given, by using two transversals that intersect exactly in one cell, and the number of such pairs of transversals in latin squares of order ≤ 5 is found. It is proved that there do not exist latin squares of order 5 with recursively differentiable prolongations, obtained by the given method, where one of the transversals is the main diagonal with the given order of elements 1,2,3,4,5 or 2,3,4,5,1.

The main solved scientific problem consists of determining necessary and sufficient conditions of recursive differentiability, of arbitrary finite order, of a class of n-ary groups, in presenting a new method of prolongation of finite binary quasigroups, by using two transversals that intersect exactly in one cell, and in characterizing the recursive differentiability of such prolongations.

The significance of theoretical and practical values of the work. The results related to the characterization of the recursive differentiability of binary and n-ary quasigroups, as well as the prolongation of quasigroups by various methods, refer to open problems in quasigroup theory related to the estimation of the spectrum of recursively differentiable quasigroups and have essential applications in the theory of MDS codes for detecting/correcting the maximum number of errors, providing new estimations of the parameters of such codes.

Publications on the topic of the doctoral thesis. The research results were published in 14 scientific works, including 3 articles in peer-reviewed journals, 2 articles in proceedings of scientific conferences, and 9 abstracts at scientific conferences.

Thesis structure and volume. The thesis is written in romanian and contains: an introduction, three chapters, general conclusions and recommendations, bibliography of 129 items, 2 annexes, 11 figures and 16 tables. The thesis has a volume of 133 pages (including 104 pages of basic text).

Keywords: quasigroup, latin square, transversal, recursive derivative, recursively differentiable quasigroup, prolongation of latin squares.

2. CONTENTS OF THE THESIS

The thesis consist of three chapters containing theoretical and practical results regarding recursively differentiable binary and n-ary quasigroups, methods of prolongation of finite quasigroups, and recursive differentiability of prolongations.

In the **Introduction**, the topicality and importance of the problem addressed is argued, the purpose and objectives of the research are formulated, the scientific novelty and originality are argued, as well as the theoretical significance and applicative value of the doctoral thesis. The scientific problem studied highlights the theoretical and applicative importance of the work. A brief analysis of the problems and publications on the topic of the thesis is presented. At the end of this section, a summary of the content of the work is presented..

The first chapter – ***Analysis of the situation in the field of recursively differentiable quasigroup theory and quasigroup extension methods*** – consists of three paragraphs. This chapter reviews the current state in the field of recursively differentiable quasigroup theory, since the emergence of this class of quasigroups within the theory of MDS codes. The study of recursively differentiable quasigroups allows, in particular, the construction of new MDS codes and the obtaining of new characterizations of the parameters of these codes (minimum Hamming distance, length, number of codewords, etc.). The relationship between the parameters of a complete recursive code, in particular, complete recursive MDS code, and the order of recursive differentiability of a finite quasigroup raises the question of the existence of finite (binary or n-ary) recursively differentiable quasigroups with a given order of recursive differentiability. Among the unsolved problems to date is the one concerning the existence of recursively 1-differentiable binary quasigroups of order 14, 18 or 26.

A prolongation of a finite quasigroup Q is a construction by which higher-order quasigroups can be obtained by adding one or more new elements to Q, and redefining the operation of this quasigroup. Several methods of prolongation of finite quasigroups are known in the literature. The study of the recursive differentiability of quasigroup prolongations offers the possibility to obtain characterizations of the spectrum (order) of recursively differentiable quasigroups. One of the existing approaches in the development of methods for constructing prolongations of finite quasigroups is the one that uses complete, respectively quasi-complete, mappings of quasigroups. Complete mappings of a finite quasigroup define transversals in the corresponding latin square. Thus, methods of prolongation of finite quasigroups, using complete substitutions, are found in combinatorics as methods of prolongation of latin squares using their transversals.

This chapter presents general results concerning complete mappings of a quasigroup and the existence of transversals in Latin squares. The known methods of prolongation of finite quasigroups (Bruck, Belousov, Belyavskaya, Dudek-Derienko, etc.) are described and illustrated by examples.

Therefore, Chapter 1 characterizes the relationships between the algebraic structure of quasigroups, coding theory, and combinatorial aspects of latin squares, providing a rigorous theoretical framework for the construction of recursively differentiable quasigroups, the construction and analysis of recursive codes with optimal parameters, and methods for constructing prolongations of finite quasigroups (latin squares).

The second chapter – ***Recursively Differentiable Quasigroups*** – consists of three paragraphs. This chapter presents algebraic properties of binary and n-ary recursively 1-differentiable quasigroups, in particular those of binary and n-ary groups. Criteria for recursive differentiability of binary quasigroups are given, characterizing the recursiv 1-differentiability of their parastrophes (Proposition 2.1.2), and relations between some groups (the multiplication group, the right multiplication group, the automorphism group, the semiautomorphism group) of a recursively 1-differentiable quasigroup and those of its recursive derivative of order 1 (Propositions 2.1.7 and 2.1.8).

Proposition 2.1.2. *Let $(Q,\cdot)$ be a binary quasigroup. The following statements hold:*

1. ${}^{l}(\cdot)$ is recursively 1-differentiable if and only if $(\cdot) \perp^{lr} (\cdot)$;

2. ${}^{r}(\cdot)$ is recursively 1-differentiable if and only if ${}^{rl}(\cdot) \perp^{lr} (\cdot)$;

3. ${}^{rl}(\cdot)$ is recursively 1-differentiable if and only if ${}^{s}(\cdot) \perp^{r} (\cdot)$;

4. ${}^{lr}(\cdot)$ is recursively 1-differentiable if and only if ${}^{l}(\cdot) \perp^{r} (\cdot)$;

5. ${}^{s}(\cdot)$ is recursively 1-differentiable if and only if $(\cdot) \perp^{rl} (\cdot)$.

Proposition 2.1.7. *Let $(Q,\circ)$ be a recursively 1-differentiable quasigroup and let $(Q,\overset{1}{\circ})$ be its recursive derivative of order one: $x \overset{1}{\circ} y = y \circ (x \circ y), \forall x, y \in Q$. Then any congruence of the quasigroup $(Q,\circ)$ is also a congruence in $(Q,\overset{1}{\circ})$.*

Proposition 2.1.8. *Let $(Q,\circ)$ be a quasigroup and let $(Q,\overset{1}{\circ})$ be its recursive derivative of order 1. The following inclusions hold:*

a) $RM\left(Q,\overset{1}{\circ}\right) \subseteq M(Q,\circ)$;

b) $Aut(Q,\circ) \subseteq Aut(Q,\overset{1}{\circ})$.

In Theorem 2.1.1, the recursive s-differentiability of the quasigroup $(\mathbb{Z}_n,*)$ is characterized, where $x * y = \bar{a}\,x + y, \forall x, y \in \mathbb{Z}_n, (a,n) = 1$, $(s \geq 1)$. Applying this theorem in Proposition

2.1.10, examples of recursive quasigroups $(q-2)$ -)-differentiable of prime order $3 \leq q \leq 19$ are obtained.

Theorem 2.1.1. *Let* $n \geq 2, a = n-k, k \in \{1, \ldots, n-1\}, (a,n) = 1$ *şi* $x * y = \overline{a}x + y, \forall x, y \in \mathbb{Z}_n$. *If, for some* $s \geq 1$, *the recursive derivatives* $(\mathbb{Z}_n, \overset{s}{*})$ *and* $(\mathbb{Z}_n, \overset{s+1}{*})$, *where* $x \overset{i}{*} y = \overline{u_i}x + \overline{v_i}y$, $i = s, s+1$, *are quasigroups, then* $(\mathbb{Z}_n, \overset{s+2}{*})$ *is a quasigroup if and only if* $(-kc_s + c_{s+1}, n) = 1$, *where* c_s *and* c_{s+1} *are the remainders from dividing* v_s *and* v_{s+1} *by* n, *respectively.*

Proposition 2.1.10. *Let* $(\mathbb{Z}_n, *)$, *where* $x * y = \overline{a}x + y, \forall x, y \in \mathbb{Z}_n$, *be a quasigroup. The following statements are true:*

1. $(\mathbb{Z}_3, *)$ *is recursively 1-differentiable if and only if* $a = 1$;
2. $(\mathbb{Z}_5, *)$ *is recursively 3-differentiable if and only if* $a = 3$;
3. $(\mathbb{Z}_7, *)$ *is recursively 5-differentiable if and only if* $a = 1$ *or* 4;
4. $(\mathbb{Z}_{11}, *)$ *is recursively 9-differentiable if and only if* $a = 3$ *or* 4;
5. $(\mathbb{Z}_{13}, *)$ *is recursively 11-differentiable if and only if* $a = 5{,}8$ *or* 11;
6. $(\mathbb{Z}_{17}, *)$ *is recursively 15-differentiable if and only if* $a = 7$ *or* 10;
7. $(\mathbb{Z}_{19}, *)$ *is recursively 17-differentiable if and only if* $a = 1{,}5$ *or* 7.

In Propositions 2.1.4 and 2.1.5, all recursively differentiable quasigroups, defined on sets of ≤ 4 elements, are determined.

Proposition 2.1.4. *There are exactly 6 recursively 1-differentiable quasigroups of order 3.*

Proposition 2.1.5. *There are exactly 48 recursively 1-differentiable quasigroups of order 4, of which 8 are recursively 2- differentiable (the maximum possible order).*

In paragraph 2.2, recursively differentiable binary groups are studied. For the criterion of recursive s −differentiability of finite abelian groups, given in [16], new equivalent conditions are presented (Proposition 2.2.2).

Proposition 2.2.1. [16] *Let* $(G, \cdot)$ *be an abelian group. For any* $\forall n \geq 3$, *and any* $x, y, a \in G$, *the following equalities hold:*

$$x \overset{n}{\cdot} a = x^{b_n} \cdot a^{b_{n+1}}; \; a \overset{n}{\cdot} y = y^{b_{n+1}} \cdot a^{b_n},$$

where $(b_n)_{n \in \mathbb{N}^*}$ *is the Fibonacci sequence:*

$$b_1 = b_2 = 1, b_k = b_{k-2} + b_{k-1}, \forall k \geq 3.$$

Proposition 2.2.2. *Let* $(G, \cdot)$ *be a finite abelian group and let* $(b_n)_{n \in \mathbb{N}^*}$ *be the Fibonacci sequence, where* $b_1 = b_2 = 1$. *The following statements are true:*

1. *The group* $(G, \cdot)$ *is recursively k- differentiable if and only if the function* $\varphi_i(x) = x^{b_i}$ *is*

injective for every $i = \overline{1, k+1}$.

2. The group $(G,\cdot)$ *is recursively* k*-differentiable if and only if* $x^{b_i} \neq e, \forall x \in G\setminus \{e\}$, $i = \overline{1, k+1}$.

3. Any finite abelian group of odd order is recursively 1- differentiable.

4. Any finite abelian group of even order is not recursively 1 - differentiable.

In the last paragraph of Chapter 2, recursively differentiable n-ary quasigroups are investigated. The main result of this paragraph is contained in Theorem 2.3.1, where a criterion for recursive r-differentiability ($r \geq 1$) of the n-ary group (Q, B) is given, where $B(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$, $(Q,\cdot)$ is a finite binary abelian group, and $n \geq 2$.

Proposition 2.3.1. *Let* $(Q,\cdot)$ *be a finite binary group and* $n \geq 2$. *The* n*-ary group* (Q, B), *where* $B(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n, \forall x_1, x_2, \ldots, x_n \in Q$, *is recursively 1-differentiable if and only if the function* $x \to x^2$ *is a bijection in* $(Q,\cdot)$.

Corollary 2.3.1. *There exist recursively 1- differentiable finite n-quasigroups of any odd order* $q \geq 3$, *for any* $n \geq 2$.

Theorem 2.3.1. *Let* $(Q,\cdot)$ *be a finite binary abelian group and let* $n \geq 2, r \geq 1$ *be two natural numbers. The* n*-ary group* (Q, B), *where* $B(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$, *for any* $x_1, x_2, \ldots, x_n \in Q$, *is recursively* r*-differentiable if and only if the functions* $x \to x^{s_i^k}$ *are bijections in the group* $(Q,\cdot)$, $\forall i = 1, \ldots, n$ *and for any* $k = 1, \ldots, r$, *where the sequences* $(s_i^k)_{k \geq 0}$ *are defined as follows:*

1. $k = 0$

$$s_1^0 = \ldots = s_n^0 = 1;$$

2. $1 \leq k < n$

$$s_t^k = s_t^0 + \ldots + s_t^{k-1},\ \forall t = 1, \ldots, k;$$

$$s_t^k = 1 + s_t^0 + \ldots + s_t^{k+1},\ \forall t = k+1, \ldots, n;$$

3. $k \geq n$

$$s_t^k = s_t^{k-n} + \ldots + s_t^{k-1},\ \forall t = 1, \ldots, n.$$

This result generalizes the criterion obtained by V. Izbash and P. Syrbu for finite abelian groups [16].

The last chapter – ***Recursive differentiability of some prolongations of quasigroups*** – consists of three paragraphs. This section is dedicated to analyzing the possibility of extending a finite quasigroup by adjoining new elements and redefining the operation, as well as studying the recursive differentiability of the prolongations.

In Paragraph 3.1, necessary and sufficient conditions are determined, such that the prolongations of a finite binary quasigroup, obtained using the R.H.Bruck's and V.Belousov's constructions, respectively, are recursively 1-differentiable (Theorems 3.1.1 and 3.1.2).

Let $(Q,\cdot)$ be a finite quasigroup of order q and $Q=\{1,2,\dots,q\}$ such that the function $x\mapsto x\cdot x$ is a bijection. Then the main diagonal of the Cayley table of $(Q,\cdot)$ is a transversal, whose elements are given by the function $\theta\colon Q\mapsto Q, \theta(x)=x\cdot x$. R.H. Bruck considered such prolongations for idempotent quasigroups, i.e., in the case $\theta=\varepsilon$, where ε is the identical substitution on Q.

According to Bruck's method, the prolongation $(Q',\circ)$ of the quasigroup $(Q,\cdot)$, where $Q=\{1,\dots,q\}$ and $Q'=Q\cup\{\xi\}, \xi\not\in Q$, is defined as follows:

$$x\circ y=\begin{cases} x\cdot y \ \ if \ \ x\neq y \ \ and \ \ x,\ y\in Q; \\ \xi \ \ if \ \ x=y \ \ and \ x\in Q; \\ \theta(x) \ \ if \ \ y=\xi \ \ and \ \ x\in Q; \\ \theta(y) \ \ if \ \ x=\xi \ \ and \ \ y\in Q; \\ \xi \ \ if \ \ x=y=\xi. \end{cases} \tag{1}$$

Thus, the prolongation $(Q',\circ)$ is a quasigroup with the Cayley table:

$\circ$	1	...	q	ξ
1	ξ	...	...	$\theta(1)$
...	...	...	...	...
q	...	...	ξ	$\theta(q)$
ξ	$\theta(1)$	...	$\theta(q)$	ξ

Table 3.1.

where $x\circ y=x\cdot y$, for any $x\neq y$ in Q. Note that not every transversal on the main diagonal gives a recursively 1-differentiable prolongation, as follows from the next statement.

Proposition 3.1.1. *Let $(Q,\cdot)$ be a finite quasigroup such that the function $\theta\colon Q\mapsto Q, \theta(x)=x\cdot x$ is a bijection. If the prolongation $(Q',\circ)$ given in (1), where $Q'=Q\cup\{\xi\}$, $\xi\not\in Q$, is a quasigroup, then $\theta(x)\neq x, \forall x\in Q$.*

Theorem 3.1.1. *Let $(Q,\cdot)$ be a finite quasigroup such that the function $\theta\colon Q\mapsto Q$, $\theta(x)=x\cdot x$ is a bijection and $\theta(x)\neq x, \forall x\in Q$. Then the prolongation $(Q',\circ)$, obtained using Bruck's method, where $Q'=Q\cup\{\xi\}$, $\xi\not\in Q$, is recursively 1-differentiable if and only if the following conditions are satisfied:*

1. *$\{f_x \mid x\in Q\}=Q$, where $f_x\cdot x=x, \forall x\in Q$;*
2. *θ is a complete substitution of $(Q,\cdot)$;*
3. *for any $x\in Q, \{\theta(x), y\cdot(x\cdot y), \theta^2(x) \mid y\in Q,\ x\neq y,\ y\neq x\cdot y\}=Q$.*

Belousov's method of prolongation uses an arbitrary transversal of the Cayley table, not necessarily one on the main diagonal. Let $\{(x, \theta(x)) \mid x \in Q\}$, where $\theta \in S_Q$, be a transversal of a finite quasigroup $(Q, \cdot)$. Then the function $\theta' \colon Q \to Q, \theta'(x) = x \cdot \theta(x)$ is a bijection. According to Belousov's method, the prolongation $(Q', \circ)$, where $Q' = Q \cup \{\xi\},\ \xi \notin Q$, is defined as follows:

$$x \circ y = \begin{cases} x \cdot y\ \ if\ \ y \neq \theta(x)\ \ and\ \ x,\ y \in Q; \\ \xi\ \ if\ \ y = \theta(x)\ \ and\ \ x,\ y \in Q; \\ \theta'(\theta^{-1}(y))\ \ if\ \ x = \xi\ \ and\ \ y \in Q; \\ \theta'(x)\ \ if\ \ y = \xi\ \ and\ \ x \in Q; \\ \xi\ \ if\ \ x = y = \xi. \end{cases} \tag{2}$$

We note that if θ' is a bijection, then $(Q', \circ)$ is a quasigroup with the following Cayley table:

$\circ$	...	$\theta(x)$	...	y	...	ξ
...	...	...	...	...	...	...
x	...	$\theta'(x)$	...	$x \cdot y$	...	$\theta'(x)$
...	...	...	...	...	...	...
ξ	...	...	...	$\theta'\big(\theta^{-1}(x)\big)$	...	ξ

Table 3.3

Proposition 3.1.2. *Let* $(Q, \cdot)$ *be a finite quasigroup and* $\theta \in S_Q$ *such that* $\theta' \colon Q \to Q$, $\theta'(x) = x \cdot \theta(x)$ *is a bijection. Then the recursive derivative* $(Q', \overset{1}{\circ})$ *of the prolongation* $(Q', \circ)$ *given in (2) is the following:*

$$x \overset{1}{\circ} y = \begin{cases} y \cdot (x \cdot y)\ \ if\ \ y \neq \theta(x \cdot y),\ y \neq \theta(x)\ \ and\ \ x,\ y \in Q; \\ \xi\ \ if\ \ y = \theta(x \cdot y),\ y \neq \theta(x)\ \ and\ \ x,\ y \in Q; \\ \theta'(\theta(x))\ \ if\ \ y = \theta(x)\ \ and\ \ x, y \in Q; \\ y \cdot \theta'(\theta^{-1}(y))\ \ if\ \ y \neq \theta(\theta'(\theta^{-1}(y))),\ x = \xi\ \ and\ \ y \in Q; \\ \xi\ \ if\ \ y = \theta(\theta'(\theta^{-1}(y))),\ x = \xi\ \ and\ \ y \in Q; \\ \theta'(\theta^{-1}(\theta'(x)))\ \ if\ \ y = \xi\ \ and\ \ x \in Q; \\ \xi\ \ if\ \ x = y = \xi. \end{cases}$$

Theorem 3.1.2. *Let* $(Q, \cdot)$ *be a finite quasigroup,* $\theta \in S_Q$ *such that the function* $\theta' \colon Q \mapsto Q$, $\theta'(x) = x \cdot \theta(x)$ *is a bijection, and let* $\theta^{-1}(y) \neq \theta'(\theta^{-1}(y)), \forall y \in Q$. *Then the prolongation* $(Q', \circ)$, *obtained by Belousov's method, is recursively 1-differentiable if and only if the following conditions hold:*

1. $\{\theta^{-1}(y)/y \mid y \in Q\} = Q$;
2. *the substitution* $y \mapsto y \cdot \theta'(\theta^{-1}(y))$ *is a bijection in* Q;
3. *for any* $x \in Q$, $\{\theta'(\theta(x)), y \cdot xy, \theta'(\theta^{-1}(\theta'(x))) | y \neq \theta(x \cdot y), y \neq \theta(x), y \in Q\} = Q$.

In Paragraph 3.2, a method is proposed for constructing prolongations of a finite quasigroup using two transversals (of the Cayley table of the quasigroup) that intersect at a single point. It is shown that there are 12 ways to extend a latin square using this method. The existence of pairs of transversals that intersect exactly at one point is analyzed, both in the general case and in latin squares of order 4 (in latin squares of order 3 there are no such pairs of transversals, as follows from p. a) below). It is shown that:

a) a latin square of order n can have at most $n-2$ transversals that intersect at a single point, being disjoint two by two at the other points (Proposition 3.2.1);

Definition 3.2.1. *We call a free transversal of a latin square* L *of order* n, *any set of* n *cells taken by exactly one from each row and each column of* L.

Proposition 3.2.1. *A latin square of order* n *can have at most* $n-2$ *ordinary transversals that intersect at a single point, being disjoint in pairs at the other points.*

b) in any latin square of order four there are exactly 96 pairs of free transversals, and respectively 13824 pairs of ordinary transversals, which intersect exactly in one cell (Proposition 3.2.3; Corollary 3.2.3).

Proposition 3.2.2. *The latin square of order four has exactly 24 free transversals.*

Proposition 3.2.3. *In any latin square of order four, there are exactly 96 pairs of free transversals that intersect exactly in one cell.*

Corollary 3.2.3. *In any latin square of order four, there are exactly 13824 pairs of ordinary transversals that intersect exactly in one cell.*

Let $(Q,\cdot)$ be a finite quasigroup of order q and let the Cayley table of the quasigroup $(Q,\cdot)$ contain two transversals T_1 and T_2, that have a single common cell, located at the intersection of the row of the element x with the column of the element y. We denote the element in the common cell of the transversals T_1 and T_2 by a. We now consider the set $Q' = Q \cup \{\xi_1, \xi_2\}$, where $\xi_1, \xi_2 \notin Q$. On the set Q' we define the operation $\circ$ as follows: we fill the cells, other than the common cell, of one transversal with ξ_1, and those of the other with ξ_2. We transfer the elements from the cells of the two transversals, except for the element in the common cell, keeping their order, to the cells of the row and column of the elements ξ_1 and ξ_2, respectively. The cells remaining in the row of the element x, the column of the element y and at the intersection of the rows and columns of the elements ξ_1 and ξ_2 can be filled in several possible ways, but following this method, there are exactly 12 possibilities for extending the quasigroup Q by adding two new elements and using 2 transversals that intersect exactly in one cell. These possibilities are illustrated in the diagrams below:

I.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	ξ_1	…	…	a	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	a	…	…	ξ_2	ξ_1
ξ_2	…	…	ξ_2	…	…	ξ_1	a

II.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	ξ_1	…	…	a	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	ξ_2	…	…	ξ_1	a
ξ_2	…	…	a	…	…	ξ_2	ξ_1

III.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	ξ_1	…	…	ξ_2	a
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	ξ_2	…	…	a	ξ_1
ξ_2	…	…	a	…	…	ξ_1	ξ_2

IV.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	ξ_1	…	…	ξ_2	a
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	a	…	…	ξ_1	ξ_2
ξ_2	…	…	ξ_2	…	…	a	ξ_1

V.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	ξ_2	…	…	a	ξ_1
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	a	…	…	ξ_1	ξ_2
ξ_2	…	…	ξ_1	…	…	ξ_2	a

VI.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	ξ_2	…	…	a	ξ_1
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	ξ_1	…	…	ξ_2	a
ξ_2	…	…	a	…	…	ξ_1	ξ_2

VII.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	ξ_2	…	…	ξ_1	a
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	ξ_1	…	…	a	ξ_2
ξ_2	…	…	a	…	…	ξ_2	ξ_1

VIII.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	ξ_2	…	…	ξ_1	a
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	a	…	…	ξ_2	ξ_1
ξ_2	…	…	ξ_1	…	…	a	ξ_2

IX.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	a	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	ξ_1	…	…	ξ_2	a
ξ_2	…	…	ξ_2	…	…	a	ξ_1

X.

$\circ$	…	…	y_0	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
x_0	…	…	a	…	…	ξ_1	ξ_2
…	…	…	…	…	…	…	…
…	…	…	…	…	…	…	…
ξ_1	…	…	ξ_2	…	…	a	ξ_1
ξ_2	…	…	ξ_1	…	…	ξ_2	a

$$\text{XI.}\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \\
\xi_2 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2
\end{array}
\qquad
\text{XII.}\quad \begin{array}{c|ccccc|cc}
\circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
x_0 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\xi_1 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \\
\xi_2 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a
\end{array}$$

Figure 3.3

Proposition 3.2.4. *Six of the 12 possible prolongations (Cases I, IV, VI, VII, X, XI), by adding two new elements and using two transversals that intersect exactly in one cell, are not recursively 1-differentiable.*

Problem 3.2.1. *Can the prolongations of a recursively 1-differentiable quasigroup, obtained by the considered method (by adding two new elements and by using two transversals that intersect exactly in one cell), be recursively 1- differentiable?*

In the last paragraph of Chapter 3, the prolongations of quasigroups of order 5 (latin squares of order 5) are studied using the proposed method, i.e., by adjoining two new elements and using two transversals that intersect exactly in a single cell.

Proposition 3.3.1. *In a latin square L of order 5 there are 45 pairs of free transversals, one of them being the main diagonal of L, each pair of transversals having a single cell in common.*

It is proved that there are exactly 48 different latin squares of order 5, which possess 2 transversals, one of which is the main diagonal of T (with the fixed order of the elements), and the second has a single cell in common with T (Proposition 3.3.2).

Proposition 3.3.2. *There are exactly 48 latin squares of order 5, which have 2 transversals that intersect in a single cell, one of these transversals being the main diagonal, with the fixed order of the element.*

Remark that the 48 Latin squares have a total of 240 pairs of transversals that intersect in a single cell, one of the transversals of each pair being on the main diagonal with the fixed order of the elements in it.

Annex 1 presents the 240 pairs of transversals (T, S) that intersect in a single cell, for the case when the order of the elements on the main diagonal T is 2,3,4,5,1.

The problem of the recursive 1-differentiability of prolongations of quasigroups of order 5, by adjoining two new elements and using two transversals that intersect exactly at one point, one of which is the main diagonal T, is solved (negatively), when the order of the elements in the transversal T is 1,2,3,4,5 or 2,3,4,5,1 (Proposition 3.3.3).

Proposition 3.3.3. *The prolongations of latin squares of order 5 by the addition of two new elements and the use of two transversals, with a single common cell, one of which is on the main diagonal and has the fixed order of elements 1,2,3,4,5 or 2,3,4,5,1, are not recursively 1-differentiable.*

3. GENERAL CONCLUSIONS AND RECOMMENDATIONS

The present thesis refers to the theory of recursively differentiable binary and n-ary quasigroups, methods of prolongation of finite quasigroups, and the study of the recursive differentiability of prolongations.

The main scientific problem solved consists of proving a criterion of recursive differentiability of arbitrary finite order for a class of n-ary groups, in presenting a new method of prolongation of finite binary quasigroups by using two transversals that intersect exactly in a cell, and in characterizing the recursive differentiability of these prolongations.

The thesis provides characterizations of the recursive differentiability of binary and n-ary quasigroups, including prolongations of quasigroups, and estimates of the maximum order of recursive differentiability and the spectrum of finite recursively differentiable quasigroups.

The present work containes criteria for the recursive differentiability of order $r \geq 1$ of binary and n-ary quasigroups, and presents a new method of prolongation of finite binary quasigroups by adding two new elements and using two transversals that intersect exactly in a cell, characterizing the number of such pairs of transversals in latin squares of order ≤ 5. The non-existence of recursively differentiable latin squares of order 5 is proved, whose prolongations obtained by the given method, one of the transversals being the main diagonal with a certain pre-established order of elements, are recursively differentiable.

The author's contribution can be summarized in ***the*** following ***main conclusions***:

1. A criterion for recursive r-differentiability ($r \geq 1$) of the n-ary group (Q, B), where $B(x_1^n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n, \forall x_1, x_2, \ldots, x_n \in Q,\ n \geq 2$, and $(Q, \cdot)$ is a finite binary abelian group, was given. This result generalizes in the n-ary case the criterion obtained by V. Izbash and P. Syrbu for finite abelian groups in [16].

2. The recursive s-differentiability of the quasigroup $(\mathbb{Z}_n, *)$ was characterized, where $x * y = \overline{a}x + y, \forall x, y \in \mathbb{Z}_n$, $(a, n) = 1, s \geq 1$. This result allows the construction of recursively $(q-2)$-differentiable linear quasigroups (over the group $\mathbb{Z}_q$).

3. All recursively differentiable quasigroups of order ≤ 4 have been determined. It is shown that there are 6 recursively 1-differentiable quasigroups of order 3, 48 recursively 1-differentiable quasigroups of order 4, and 8 recursively 2-differentiable quasigroups of order 4 (Propositions 2.1.4 and 2.1.5).

4. A method for constructing prolongations of a finite quasigroup, by adjoining two new elements and using two transversals (of the Cayley table of the quasigroup) that intersect at a single

point, was proposed. The existence of such pairs of transversals was analyzed. In particular, it was shown that:

a) a latin square of order n can have at most $n - 2$ ordinary transversals that intersect at a single point, being disjoint two by two at the other points;

b) in any latin square of order four there are exactly 96 pairs of free transversals, and respectively 13824 pairs of ordinary transversals, which intersect exactly in one cell;

c) there are exactly 48 different latin squares of order 5, each with 2 transversals, one of which is the main diagonal T (with the fixed order of elements), and the second has only one cell in common with T; there are exactly 240 pairs of transversals corresponding to the 48 different latin squares of order 5.

5. The problem of recursive 1-differentiability of prolongations of quasigroups of order 5, obtained by adjoining two new elements and using two transversals that intersect exactly at one point, one of which is the main diagonal T, when the order of the elements in T is 1, 2, 3, 4, 5 or 2, 3, 4, 5, 1, has been solved (negatively) .

6. Necessary and sufficient conditions, such that the prolongations of finite binary quasigroups, obtained using the Bruck and Belousov constructions, respectively, are recursively 1-differentiable, were determined.

The author's results related to the thesis topic are published in [1-14].

The thesis proposed for defense contains criteria for recursive differentiability of finite binary and finite n-ary quasigroups, and of the prolongations of quasigroups by various methods. The thesis proposes and studies a new method of prolongation of finite quasigroups by adjoining two new elements and using two transversals that intersect in a single cell.

Recommendations:

a) The proposed method for extending finite quasigroups can be generalized for any suitable number of transversals of a latin square that intersect in a single cell.

b) The results concerning the recursive 1-differentiability of the prolongations can be used to characterize their recursive differentiability of order $r \geq 2$.

c) The conditions and criteria for the recursive differentiability of finite binary or n-ary quasigroups (groups) can be applied to obtain new estimations of the spectrum of these quasigroups (groups).

d) The characterizations related to the new method of prolongation of quasigroups, presented in the present thesis, can serve as a tool for researching the existence of such recursively differentiable prolongations. In particular, the existence of prolongations of the given type, which are recursively 1-differentiable, in the case of quasigroups of order 5 (general case) remains an open problem.

e) The results of the thesis can be used for further research in the field of quasigroup theory and in adjacent fields of algebra, geometry, and combinatorics, in code theory and cryptography. The results can also be used as support for specialized university courses.

BIBLIOGRAPHY

1. ABASHIN, A. Linear recursive MDS-codes of dimension 2 and 3. In: *Discret. Mat.* 12 (1998), pp. 140 – 153. ISSN 0924-9265
2. BELOUSOV, V. Extensions of quasigroups. In: *Izv. Akad. Nauk Mold. SSR, Ser. Fiz.-Tekh. Mat. Nauk*, No. 8 (1967), pp. 3 – 24. (in Russian)
3. BELOUSOV, V. *Foundations of the Theory of Quasigroups and Loops*. Nauka, Moscow, 1967. (in Russian) 224 p.
4. BELOUSOV, V. D., SANDIK, M. D. n-ary Quasi-groups and Loops. *Sib. Math. J.* 7(1) (1966), pp. 24 – 42. ISSN 0037-4466
5. BELYAVSKAYA, G. B. Contraction of quasigroups. I. In: *Bul. Akad. Stiince RSS Moldoven*, (1), 1970, pp. 6 – 12. (in Russian)
6. BELYAVSKAYA, G. B. Contraction of quasigroups. II. In: *Bul. Akad. Stiince RSS Moldoven,* (3), 1970, pp. 3 – 17. (in Russian)
7. BELYAVSKAYA, G. B. Generalized extension of quasigroups. In: *Mat. Issled.*, 5(2), 1970, pp. 28 – 48. (in Russian)
8. BELYAVSKAYA, G. B. Recursively r-differentiable quasigroups within S-systems and MDS-codes. In: *Quasigroups Relat. Systems* 20 (2012), pp. 157 – 168. ISSN 1561-2848
9. BRUCK, R. H. Some results in the theory of quasigroups. In: *Trans. Amer. Math. Soc.*, 55 (1944), pp. 19 – 52. ISSN 0002-9947
10. COUSELO, E., GONZALEZ, S., MARKOV, V., NECHAEV, A. Recursive MDS codes and recursively differentiable quasigroups. In: *Discrete Math. Appl*. 8, No. 3 (1998), pp. 217 – 245. ISSN 0234-0860
11. DERIYENKO, I. I., DUDEK, W. A. Contractions of quasigroups and Latin squares. In: *Quasigroups Relat. Systems* 21 (2013), pp. 165 – 174. ISSN 1561-2848
12. DERIYENKO, I. I., DUDEK, W. A. On prolongations of quasigroups. In: *Quasigroups Related Systems* 16 (2008), pp. 187 – 198. ISSN 1561-2848
13. EVANS, A. B. The existence of complete mappings of finite groups. In: *Congr. Numer*. 90 (1992), pp. 65 – 75. ISSN 0384-9864
14. EVANS, A. B. The existence of complete mappings of SL(2,q), q≡3 modulo 4. In: *Finite Fields Appl*. 11 (2005), pp. 151 – 155. ISSN 1071-5797
15. IZBASH, V., SYRBU, P. On recursively differentiable binary quasigroups. In: *Proceedings of the 11-th Conf. on Applied and Industrial Mathematics (CAIM2003),* May 29 – 31, 2003, Oradea, Romania, Vol. 1, pp. 149 – 152. ISBN 973-613-330-3

16. IZBASH, V., SYRBU, P. Recursively differentiable quasigroups and complete recursive codes. In: *Commentat. Math. Univ. Carol.* 45, No.2, (2004), pp. 257 – 263. ISSN 0010-2628
17. KEEDWELL A. D., DENES, J. *Latin Squares and their Applications,* Second Edition. Elsevier Science, 2015, 438 p. ISBN 978-0-444-63555-6
18. KEEDWELL A. D., DENES, J. *Latin Squares and their Applications*. Akadémiai Kiadó, Budapest; Academic Press, New York; English Universities Press, London, 1974. 547p. ISBN 978-0122-09-350-0
19. LARIONOVA, I., SYRBU, P. On Recursive Differentiability of Binary Quasigroups. In: *Studia Universitatis Moldaviae*, 2 (82), 2015, pp. 53 – 60. ISSN 1857-2073
20. MANN H. The construction of orthogonal latin squares. In: *Ann. Math. Stat.*, 13, 1942, pp. 418-423. ISSN 0003-4851
21. MARKOV, V., NECHAEV, A., SKAZHENIK, S., TVERITINOV, E. Pseudogeometries with clusters and an example of a recursive $[4,2,3]_{42}$-code. In: *Fundam. Prikl. Mat.*, 14(4), 2008, pp. 181 – 192. ISSN 1560-5159
22. PAIGE, L. J. Complete mappings of finite groups. In: *Pacific J. Math.*, 1(1) (1951), pp. 111 – 116. ISSN 0030-8730
23. PFLUGFELDER, H. O. *Quasigroups and loops: introduction*. Sigma Series in Pure Mathematics, 7. Heldermann Verlag, Berlin, 1990. 147 p. ISBN 978-3885-38-007-8
24. SYRBU, P. On the order of recursive differentiability of finite binary quasigroups. In: *Bul. Acad. Ştiinţe Repub. Mold. Mat.*, No. 3 (103) (2023), pp. 103 – 106. ISSN 1024-7696
25. WANLESS, I. M. A generalisation of transversals for latin squares. In: *Electron. J. Combin.* 9(1) (2002), R12. ISSN 1077-8926
26. WANLESS, I. M. Transversals in latin squares. In: *Quasigroups Related Systems* 15 (2007), pp. 169 – 190. ISSN 1561-2848
27. XU, L., BRUCK, J. X-code: MDS array codes with optimal encoding. In: *IEEE Transactions on Information Theory*, 45(1) (1999), pp. 272 – 276. ISSN 0018–9448
28. YAMAMOTO, K. Generation principles of Latin squares. In: *Bull. Inst. Internat. Statist.*, 38: pp. 73–76, 1961.

LIST OF THE AUTHOR'S PUBLICATIONS ON THE THESIS TOPIC

Articles in journals from the SCOPUS database

1. SYRBU P., CUZNEȚOV El. On recursive 1-differentiability of the quasigroups prolongations. In: *Bul. Acad. Ştiințe Repub. Mold. Mat.*, 2023, No. 2 (101), 2023, pp. 102 – 109. ISSN 1024-7696

2. SYRBU P., CUZNEȚOV El. On recursively differentiable k-quasigroups. In: *Bul. Acad. Ştiințe Repub. Mold. Mat.*, 2022, No. 2 (99), 2022, pp. 68 – 75. ISSN 1024-7696

Articles in journals from the National Register of specialized journals (category B, ANACEC)

3. CUZNEȚOV El. On a quasigroup prolongation and its recursive differentiability. In: *Acta et Commentationes Exact and Natural Sciences,* vol. 20, no. 2, 2025, pp. 25 – 39. ISSN 2537-6284

Contributions (articles, abstracts, theses) in conference proceedings and other scientific events

a. Works published in collections of articles

4. CUZNEȚOV El. On a method of prolongation of quasigroups. In: *Proceedings of the International Conference dedicated to the 60th anniversary of the foundation of Vladimir Andrunachievici Institute of Mathematics and Computer Science,* October 10 – 13, 2024, Chișinău, Republic of Moldova, pp. 55 – 60. ISBN 978-9975-68-523-8

5. CUZNEȚOV El., KUZNETSOV Eu. General Boolean Algebras, Boolean Rings and Extension Algebras. In: *Proceedings of the 5th Conference on Mathematical Foundations of Informatics*, July 3 – 6, 2019, Iași, Romania, pp. 181 – 186. ISBN 978-606-714-481-9

b. Abstracts at specialized scientific conferences

6. CUZNEȚOV El., SYRBU P. On a prolongation of quasigroups using two transversals. In: *Abstracts of the 32nd International Conference on Applied and Industrial Mathematics (CAIM-2025),* September 18 – 21, 2025, Bucharest, Romania, pp. 97 – 98. ISSN 2537-2688

7. CUZNEȚOV El. Prolongation and recursive differentiability of quasigroups. In: *Abstracts of the National conference with international participation: Natural Sciences in the Dialogue of Generations,* State University of Moldova, September 18 – 19, 2025, Chișinău, Republic of Moldova, p. 216. ISBN 978-9975-62-898-3

8. CUZNEȚOV El., SYRBU P. On recursive differentiability of quasigroups prolongations. In: *Abstracts of the International Conference on Quasigroups and Related Systems (ConfQRS 2025),* July 2 – 4, 2025, Chișinău, Republic of Moldova, pp. 9 – 10. ISBN 978-9975-62-880-8

9. CUZNEȚOV El., SYRBU P. On a new quasigroup prolongation and its recursive differentiability. In: *Abstracts of the International Conference "Mathematics & IT: Research and Education" (MITRE–2025),* June 26 – 29, 2025, Chișinău, Republic of Moldova, pp. 15 – 16. ISBN 978-9975-62-879-2 (PDF)

10. CUZNEȚOV El. Recursively differentiable finite quasigroups. In: *Abstracts of the National conference with international participation: Natural Sciences in the Dialogue of Generations,* State University of Moldova, September 12 – 13, 2024, Chișinău, Republic of Moldova, p. 262. ISBN 978-9975-62-756-6

11. CUZNEȚOV El., SYRBU P. On recursive differentiability of Bruck-Belousov prolongations of quasigroups. In: *Abstracts of the 30th Conference on Applied and Industrial Mathematics (CAIM-2023)*, Iași, Romania, September 14 – 17, 2023, p. 71. ISSN 2537-2688

12. CUZNEȚOV El., SYRBU P. On recursive differentiability of some quasigroups prolongations. In: *Abstracts of the International Scientific Conference "Mathematics & IT: Research and Education (MITRE-2023),* June 26 – 29, 2023, Chișinău, Republic of Moldova, pp. 22 – 23. ISBN 978-9975-62-535-7

13. CUZNEȚOV El., SYRBU P. On recursively differentiable n-quasigroups. In: *Abstracts of the 29th Conference on Applied and Industrial Mathematics (CAIM-2022),* August 25 – 27, 2022, Chișinău, Republic of Moldova, pp. 161 – 163. ISBN 978-9975-76-401-8

14. CUZNEȚOV El., SYRBU P. On recursively differentiable quasigroups. In: *Abstracts of the International Virtual Conference "Mathematics & IT: Research and Education (MITRE-2021),* dedicated to the 75th anniversary of Moldova State University, July 1 – 3, 2021, Chișinău, Republic of Moldova, pp. 47 – 48. ISBN 978-9975-158-19-0

ADNOTARE

la teza cu titlul ”**Extensibilitatea și derivabilitatea recursivă a quasigrupuilor”,** înaintată de către candidatul – **Cuznețov Elena,** pentru conferirea titlului științific de doctor în științe matematice la specialitatea - **111.03. Logică matematică, algebră și teoria numerelor, Chișinău, 2026.**

Structura tezei: teza este scrisă în limba română și constă din introducere, trei capitole, concluzii generale și recomandări, bibliografie din 129 de titluri și 2 anexe. Teza conține 104 de pagini cu text de bază, 11 figuri și 16 tabele. Rezultatele obținute sunt publicate în 14 lucrări.

Cuvinte-cheie: quasigrup, pătrat latin, transversală, derivată recursivă, quasigrup recursiv derivabil, prelungire a pătratului latin.

Scopul lucrării: caracterizarea derivabilității recursive a quasigrupurilor binare și n-are, inclusiv a prelungirilor quasigrupurilor, estimarea ordinului maximal de derivabilitate recursivă și a spectrului quasigrupurilor recursiv derivabile finite.

Obiectivele cercetării: caracterizarea derivabilității recursive a quasigrupurilor binare și a grupurilor n-are, determinarea unor noi metode de prelungire a quasigrupurilor binare finite și studiul derivabilității recursive ale prelungirilor quasigrupurilor.

Noutatea și originalitatea științifică: În lucrare sunt determinate criterii ale derivabilității recursive de ordin $r \geq 1$ ale quasigrupurilor binare și n-are, este data o metodă nouă de prelungire a quasigrupurilor binare finite prin utilizarea a două transversale care se intersectează exact într-o celulă, fiind caracterizat numărul de astfel de perechi de transversale în pătratele latine de ordin ≤ 5. Este demonstrată inexistența pătratelor latine de ordinul 5 cu prelungiri recursiv derivabile, obținute prin metoda dată, una dintre transversale fiind diagonala principală cu ordinea prestabilită a elementelor 1,2,3,4,5 sau 2,3,4,5,1.

Problema științifică importantă soluționată: constă în determinarea condițiilor necesare și suficiente de derivabilitate recursivă de ordin arbitrar finit a unei clase de grupuri n-are, în prezentarea unei metode noi de prelungire a quasigrupurilor binare finite prin utilizarea a două transversale care se intersectează exact într-o celulă și caracterizarea derivabilității recursive a acestor prelungiri.

Semnificația teoretică și valoarea aplicativă a lucrării: Rezultatele ce țin de caracterizarea derivabilității recursive de ordin arbitrar r ale unor clase de quasigrupuri binare și n-are, precum și a prelungirilor quasigrupuilor prin diferite metode, se referă la probleme deschise din teoria quasigrupurilor ce țin de estimarea spectrului quasigrupurilor recursiv derivabile și au aplicări esențiale în teoria MDS-codurilor pentru detectarea/corectarea unui număr maximal de erori, oferind noi estimări ale parametrilor acestor coduri

Implementarea rezultatelor științifice: Quasigrupurile recursiv derivabile n-are, $n \geq 2$, pot fi utilizate drept funcții de control la construirea MDS-codurilor și la estimarea parametrilor acestor coduri. De asemenea, caracterizarea numărului perechilor de transversale ale unui pătrat latin, care se intersectează exact într-o celulă, oferă soluții prntru probleme analogice din combinatorică. Rezultatele lucrării pot fi utilizate în calitate de suport pentru cursuri universitare de specialitate.

ANNOTATION

of the thesis entitled ”**Extensibility and recursive differentiability of quasigroups**”,
presented by the candidate **Cuznețov Elena**, for obtaining the degree of Doctor in Mathematical Sciences with specialty – **111.03. Mathematical logic, algebra and number theory,**
Chisinau, 2026.

Structure of the thesis: the thesis is written in Romanian and consists of an introduction, three chapters, general conclusions and recommendations, a bibliography of 129 titles and 2 appendices. The thesis contains 104 pages of basic text, 11 figures and 16 tables. The obtained results were published in 14 papers.

Keywords: quasigroup, Latin square, transversal, recursive derivative, recursively differentiable quasigroup, prolongation of Latin squares.

Research purpose: The purpose of the thesis is to characterize the recursive differentiability of binary and n-ary quasigroups, including prolongations of quasigroups, to estimate the maximal order of recursive differentiability and the spectrum of finite recursively differentiable quasigroups.

Research objectives: characterizing the recursive differentiability of binary quasigroups and n-ary groups, determining new methods of prolongation of finite binary quasigroups, and studying the recursive differentiability of quasigroup prolongations.

Scientific novelty and originality: In the present work, criteria for recursive differentiability of order $n \geq 1$ of binary and n-ary quasigroups are determined, a new method of prolongation of finite binary quasigroups is given, by using two transversals that intersect exactly in one cell, and the number of such pairs of transversals in Latin squares of order ≤ 5 is characterized. It is proved that there do not exist Latin squares of order 5 with recursively differentiable prolongations, obtained by the given method, where one of the transversals is the main diagonal with the order of elements 1,2,3,4,5 or 2,3,4,5,1.

The main scientific problem solved: consists in determining necessary and sufficient conditions of recursive differentiability of arbitrary finite order of a class of n-ary groups, in presenting a new method of prolongation of finite binary quasigroups, by using two transversals that intersect exactly in one cell, and in characterizing the recursive differentiability of such prolongations.

The significance of theoretical and practical values of the work: The results related to the characterization of the recursive differentiability of binary and n-ary quasigroups, as well as the extensions of quasigroups by various methods, refer to open problems in quasigroup theory related to the estimation of the spectrum of recursively differentiable quasigroups and have essential applications in the theory of MDS codes for detecting/correcting the maximum number of errors, providing new estimations of the parameters of such codes.

Implementation of the scientific results: Recursively differentiable n-ary quasigroups, $n \geq 2$, can be used as control functions in the construction of MDS codes and for the estimation of the parameters of these codes. The characterization of the number of pairs of transversals of a Latin square that intersect exactly in one cell, provides solutions for analogous problems in combinatorics. The results of the thesis can be used as a support for specialized university courses.

АННОТАЦИЯ

к диссертации **«Продолжение и рекурсивная дифференцируемость квазигрупп»**,
представленная **Кузнецовой Еленой** на соискание степени доктора математических наук
по специальности – **111.03. Математическая логика, алгебра и теория чисел,**
Кишинёв, 2026.

Структура диссертации: диссертация написана на румынском языке и состоит из введения, трех глав, общих выводов и рекомендаций, библиографии из 129 названий и 2 приложений. Диссертация содержит 104 страницы основного текста 16 таблиц и 11 рисунков. Полученные результаты опубликованы в 14-и научных.

Ключевые слова: Квазигруппа, латинский квадрат, трансверсаль, рекурсивная производная, рекурсивно дифференцируемая квазигруппа, продолжение латинского квадрата.

Цель работы: Целью диссертационной работы является характеристика рекурсивной дифференцируемости бинарных и n-арных квазигрупп, включая продолжение квазигрупп, оценка максимального порядка рекурсивной дифференцируемости и спектра конечных рекурсивно дифференцируемых квазигрупп.

Задачи работы: характеристика рекурсивной дифференцируемости бинарных и n-арных квазигрупп, нахождение новых методов продолжения конечных бинарных квазигрупп и исследование рекурсивной дифференцируемости продолжений квазигрупп.

Научная новизна и оригинальность: В работе установлены критерии рекурсивной дифференцируемости порядка $n \geq 1$ бинарных и n-арных квазигрупп, а также предложен новый метод продолжения латинских квадратов с помощью двух трансверсалей, пересекающихся точно в одной клетке, и охарактеризовано число таких пар трансверсалей в латинских квадратах порядка ≤ 5. Показано несуществование латинских квадратов 5-го порядка с рекурсивно дифференцируемыми продолжениями, полученными данным методом, где одна из трансверсалей является главной диагональю с порядком элементов 1,2,3,4,5 или 2,3,4,5,1.

Решенная важная научная задача: состоит в нахождение необходимых и достаточных условий рекурсивной дифференцируемости произвольного конечного порядка класса n-арных групп, в представлении нового метода продолжения конечных бинарных квазигрупп с помощью двух трансверсалей, пересекающихся точно в одной точке (клетке), и в описании рекурсивной дифференцируемости этих продолжений.

Теоретическая значимость и прикладная ценность работы: Результаты, связанные с описанием рекурсивной дифференцируемости некоторых классов бинарных и n-арных квазигрупп, а также построение продолжений квазигрупп различными методами, относятся к открытым задачам теории квазигрупп, связанных с оценкой спектра рекурсивно дифференцируемых квазигрупп, и имеют приложения в теории МДР-кодов для обнаружения/исправления максимального числа ошибок, позволяя получать новые оценки параметров таких кодов.

Внедрение научных результатов: Рекурсивно дифференцируемые n-квазигруппы, $n \geq 2$, могут быть использованы в качестве контрольных функций при построении МДР-кодов и оценке их параметров. Характеризация числа пар трансверсалей латинского квадрата, пересекающихся точно в одной ячейке, предоставляет решения для аналогичных задач в комбинаторике. Результаты работы могут быть использованы в качестве материала для специализированных университетских курсов.

CUZNEȚOV Elena

EXTENSIBILITY AND RECURSIVE DIFFERENTIABILITY OF QUASIGROUPS

111.03. Mathematical logic, algebra and number theory

Abstract of the doctoral thesis in mathematical sciences

Approved for printing: 09.04.2026
Offset paper. Offset printing.
Print sheets: 2.0

Paper format: 60×84 1/16
Print run: ___ **copies.**
Order no. ______

Tipografia Casa Presei „Tipocart”
str. Puşkin 22, of. 523, Chișinău, MD-2012